Is arithmetic best understood in terms of structures?

Structuralism is the view that arithmetic is not the study of a family of numerical objects, which have relations because of their intrinsic natures, but that arithmetic is only concerned with a framework of relations and resultant structures. If there are numerical objects, then only their position in the structure is significant.

This view developed during the last century, largely through developments within mathematics. A new interest in axioms led to increasingly high levels of abstraction, as an overview of mathematical practice was developed. As this happened, the startling discovery was that there might not only be several forms of a subject such as geometry, depending on which axioms were chosen, but that there could even be rival models of the same form of the subject, such as the von Neumann and Zermelo accounts of numbers in terms of sets (von Neumann says 3 is part of 4; Zermelo disagrees). The first proposal to make relations central came from Dedekind (1888:§74), supported by Hilbert and Poincaré. Within philosophy, Benacerraf gave impetus to the new approach (1965), but Nicolas Bourbaki (1950) had already collectively articulated a full structuralist view from within mathematics, claiming that mathematics centres on a 'storehouse' of three main structures – the algebraic, the ordered, and the topological (MacBride 2005:79; Shapiro 1997:176).

Shapiro's favoured analogy is a baseball defence (1997:76), where the structure and relations of the players' positions seem to form an abstract entity, which can be discussed independently of the individuals who implement it, and Benacerraf compares numbers to markings on a ruler, which only have meaning in that context (1965:292). Benacerraf asserts that '6' is the name of a *role* in the number structure. Thus the thing to understand about 6 would be that it is the successor of 5, that its successor is 7, that it is composed of 3 2's, and so on. This makes 6 a location in a network, and so any entity could fulfil the role. Thus Hilbert was heard to speculate that a beer mug might do the job, and Shapiro suggests that 6 can play the role of 3 when it is in the structure of even numbers (1997:100). It remain opens whether we are dealing with the same 6 when it features in the natural, or the rational, or the real numbers.

While Shapiro takes the arithmetical structures to nest within one another, as parts of a single structural entity, Resnik takes us to be dealing with many heterogeneous structures, referred to as 'patterns'. Resnik even suggests a structural account of set theory, which would otherwise remain a rival to the structural account (1999:218; also Hellman 2007:§2); he also suggests that proofs have a distinctive pattern (1999:9); and he proposes three types of relations to connect the various patterns together (1999:209).

Shapiro's main explanatory tool is model theory. Since mathematicians are only interested in differences between models 'up to isomorphism' (the point where two models map one-to-one, satisfying the same sets of sentences, and preserving all the functions and relations), Shapiro takes these isomorphic models to be the subject-matter of arithmetic (1997:55). For example, arithmetic can be axiomatised in ways that vary in strength, from what Smith (2007) calls 'baby arithmetic', through intermediate 'Robinson arithmetic', to the very comprehensive Peano Arithmetic (PA). Each of these systems is nested within a more comprehensive model (so an 'injective function' will map the model one-to-one into the more comprehensive model), so that the whole resulting theory is 'categorical'. Given this sort of mapping between models, Shapiro takes it that 6 is the identical structure-location as part of the natural numbers, and as part of the larger collection of real numbers (2000:267), whereas Friend denies this (on the plausible ground that they have different immediate predecessors (i.e. different relations) in the two systems) (2007:93).

Other options for structuralists that are noted by Reck and Price (2000) are a very 'formalist' version, which eschews semantics, and a 'relativist' version (Quine 1969), which just chooses *any* model to work with, without interest in super-models or 'right' models. Both Shapiro and Resnik offer versions of what Hellman calls 'hyperplatonism' (2007:542), because they are committed to the existence of one or more vast abstract structures.

An obvious attraction of these views is that they by-pass many philosophical difficulties, while conforming well to actual mathematics. Instead of having to choose which of two set theories will express our numbers for us, Benacerraf says we can view them as two descriptions of a single structure. The idea that numbers are ‘objects’, while supported by persuasive arguments from Frege, has inherent problems. Frege takes each number to have its own distinctive characteristic (1884:§10), but it is not clear what the characteristics of six could be if one were to remove all of its relations with other numbers. Thus a platonist ‘object’ is rather perplexing, and nominalists are keen to wield Ockham’s Razor, and banish them on metaphysical grounds. The fact that mathematicians lose interest in distinctions between models once isomorphism is achieved points to their shared pattern as the target, rather than the constants and terms in each model.

For some the attraction of Structuralism is the glimpse of a more unified account of reality, given that relations, functions and structures are found in both the physical and the abstract realms (symmetry, for example, or ancestry, or iteration). Quine, as part of his univocal account of ‘existence’, and his uniform ‘web of belief’ about reality, expressed sympathy with a ‘sweeping’ and ‘global’ structuralism that applied ‘to concrete and abstract objects indiscriminately’ (1992:6). Michael Jubien distinguishes some models in set theory as being ‘fundamental’, because they are based on concrete objects (1977). Zermelo-Fraenkel set theory contains no such objects, being merely an axiomatisation of the relationships between the sets themselves, founded on the empty set. Quine takes the ZF set structure to be part of our ontological commitment when we do science, but there is also a commitment to the concreta which can be theoretically structured by the sets. A unification of the two in a more ‘fundamental’ model seems appealing. Shapiro observes that “some structures are exemplified by both systems of abstracta and systems of concreta” (1997:248), and is clearly struck by the possibility that arithmetic and physical science might be parts of a single story of nature. The appeal of this view will be greater for those (such as myself) who already take a naturalistic view of mind, universals and values.

A recurrent difficulty in all philosophical accounts of arithmetic is how to deal with infinities. Most axiomatisations of arithmetic (e.g. PA) require an infinite supply of objects (of some sort), because otherwise no model of the natural numbers is possible, and so every sentence and the negation of every sentence become vacuously true (Shapiro 1997:86). An axiom asserting an infinity of physical objects (even space-time points) is wishful thinking, an infinity of mental objects is implausible, and an infinity of abstracta seems to concede the game to Frege. Structuralism seems to offer more hope, because if a recurrent pattern is correctly observed, it is not necessary to observe all of its occurrences. We can say that whenever the pattern is needed, it will be available.

However, one difficulty, found in the problem of induction, and in Wittgenstein’s scepticism about rule-following, and hence patterns(1953:§201-2), is that we cannot be sure of arithmetical structures beyond what has been observed. Thus we are not sure whether Goldbach’s Conjecture is true, or that the sequence of prime pairs continues indefinitely, despite huge numbers of instances. One response which has been explored is that arithmetic be founded on modal claims, so that we can rely on the *possibility* of various structures and a *possible* infinity of objects, without commitment to their actuality (Putnam 1967; Hellman 1989). Shapiro’s objection to this approach is that possible structures don’t seem to be very helpful in explaining the foundations of arithmetic, since they are only defined by apparent absence of contradiction (1997:229). What we want to know about are *actual* structures. However, the claims for actual structures seem to presuppose the existence of the very objects which are being challenged, and we can learn as much on a theoretical level from possible structures as from actual structures (from possible skyscrapers, for example).

A further response to the difficulty of finding an infinite supply of entities is offered by Chihara (2003), who boldly proposes that our logic should embrace a new ‘constructability quantifier’ (2004:Ch.7), which asserts of an open sentence that it can be legitimately constructed. This frees us to construct any very large cardinals that are required, without commitment to their prior existence. If the platonist pre-existent infinity of objects is rejected, it is helpful to have these nominalist accounts of how either there is no limit to constructions of objects when required, or else (with Hellman) no logical limitations on their possible existence.

Perhaps the strongest objection to the whole structuralist enterprise was first articulated by Frege (1884:§42) and Russell (1903:§242). It is that while objects in the world may exhibit order, their order is not intrinsic (even in a baseball defence, where the structure is imposed by a coach, not by the characteristics of the players). One could arrange the natural numbers in any order, but they have a *correct* order, which presumably results from their intrinsic character. If you remove the objects, it is not clear what the ordering consists of. It is not enough to say that the successor relation will do the job, because it is essential to 6 that it be the successor of *5*, and not just the successor of *something*. Dummett adds that it is more plausible to fix 6 by its place in *counting*, which seems to be based on one-to-one correspondence rather than on structural features, and also that the apparently significant debate over whether to start the natural numbers with 0 or 1 would become an empty question if only position mattered (1991:53). Indeed, it looks as if the same 6 occupies different positions in different systems that have the same structure (such as the natural numbers and the positive integers) (Dummett 1998:162). Burgess objects that mathematicians study actual separate structures, rather than what structures have in common (1997:80), but this seems to miss Shapiro's distinction between systems and structures (since mathematicians are interested in isomorphic sets of models, as well as the component sub-models). More interestingly, Burgess challenges the account of ordering (1997:86) by demanding an answer to Van Inwagen's Question – are the relations involved 'internal', 'external', or 'intrinsic'? Either you end up with objects with intrinsic relations, or you have relations, but no idea of what is being related. The latter point leads to Hersh's objection that a dressmaker studies print 'patterns' (involving colours as well as geometry), so structuralism is 'over-inclusive' if we can't say what is distinctive of *mathematical* patterns (1997:178).

All of these objections imply that only a commitment to numerical objects can distinguish what is mathematical, can show that ordering is intrinsic to the subject-matter, can show why 0 is quite distinct from 1, and can give us some idea of *what* is being structured. You would hardly understand a baseball defence if you don't know what a baseball player was. Further difficulties that have been raised include the question of the extent of the world of structures, with the whole structural or pattern system appearing to be itself a pattern or structure, which would imply a regress (Friend 2007:88), and MacBride's point that if numbers are positions in structures, then we seem to need 'positions' in our ontology, which may be even more intractable than 'objects' (2007:585). Parsons adds the difficulty that the infinite supply of entities (of some sort) must support all kinds of transfinite cardinals (in higher-order set theory), so that a mere ω-sequence of objects will not suffice (1990:331).

The issues surrounding these problems are often complex and technical (Horsten 2007:5.2 outlines the difficulties of second-order logic which face modal structuralists). There seem to be certain defences of structuralism, however, which are promising. The central objection - that structure must be ***of*** something - is powerful. If, for example, the physical world has a structure, it will arise from the world itself, and is not conjured up by our creation of relational concepts. Dummett's point that counting is prior to structures seems right. Of course, Dummett thinks the items counted are Fregean objects – roughly, sets which are the extensions of second-order properties - but my sympathies are more with Quine and Shapiro, that there must be a continuity with the physical world, and with the probable origins of our arithmetic. We cannot be simplistic here, because the 'pattern' exhibited by six physical objects will be quite different if they are arranged linearly, or in a 3x2 array, or scattered across the universe, and to spot the 6-pattern in all of those would require a *prior* knowledge of what '6' meant (thus Brown wonders whether we might need an a priori notion of a structure before we could spot one in reality (1999:59)). But I take the beautiful applicability of arithmetic to be much better explained by it being rooted in the world, than merely (with Frege) by its extreme generality.

At least a few so-called 'objects' in mathematics are just useful fictions. A well-known example is the idea of a 'limit', which offered an alternative to difficulties over the nature of 'infinitesimals'. Using that observation, a mixed strategy looks more promising. If we say that we observe patterns (think of Hume's 'resemblance' and 'contiguity' as the beginnings of an explanation), and that we notice common features in the patterns of a line of six objects and a 3x2 grid of them (notably the possibilities of one-to-one mapping, and of rearrangement), we have already gone modal when we contemplate such possibilities. At the other end of the theory, we can count until we run out of objects, and then declare further claims as either meaningful but false, or as modal assertions. To spot a pattern is, after all, to notice the *possibility* of its continuation. Thus a structuralist model of a mathematics which starts from what is clearly actual or clearly possible, and only adds fictions and more remote possibilities when it is convenient (e.g. when faced with the equation $x^2 = -1$) – which looks very like the actual practice of mathematicians – gives us a paradigm, probably the best, for the philosophy of arithmetic. This differs from Hellman in that the structures are not mere fictions or speculations, but start as features of experience. We must attend to Frege's point that we count concepts and mental events as well as physical objects, but the empirical notion of 'experience' was never confined to the mere five senses. In this way, while a hyperplatonist structuralism is excessive and metaphysically implausible, a more nominalist view is well worth exploring, especially if it focuses on the possible patterns shared by the physical and the arithmetical.

2495 words

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